$$d_{n+1}(\sigma) = \Phi\, \beta_{n+1} \Psi(\sigma) =$$

$$= \Phi\, j_n \circ \partial_{n+1} (f_\sigma)_* ([B_\sigma^{n+1}]) =$$

$$= \Phi\Big(j_n\, (f_{\partial\sigma})_* ([\partial B_\sigma^{n+1}])\Big)$$

takes you to hom of the pair

$$= \sum_{\tau \in I_n} \phi_n \Big((p_\tau)_* \big(j_n ((f_{\partial\sigma})_* [\partial B_\sigma^{n+1}]) \big) \Big) \tau$$

collapses $(n-1)$-skeleton

$$= \sum_\tau \phi_n \Big((\underbrace{p_\tau \circ f_{\partial\sigma}}_{(p_\tau)_* \circ j_n \circ (f_{\partial\sigma})_*})_* [\partial B_\sigma^{n+1}] \Big) \tau$$

here we just use

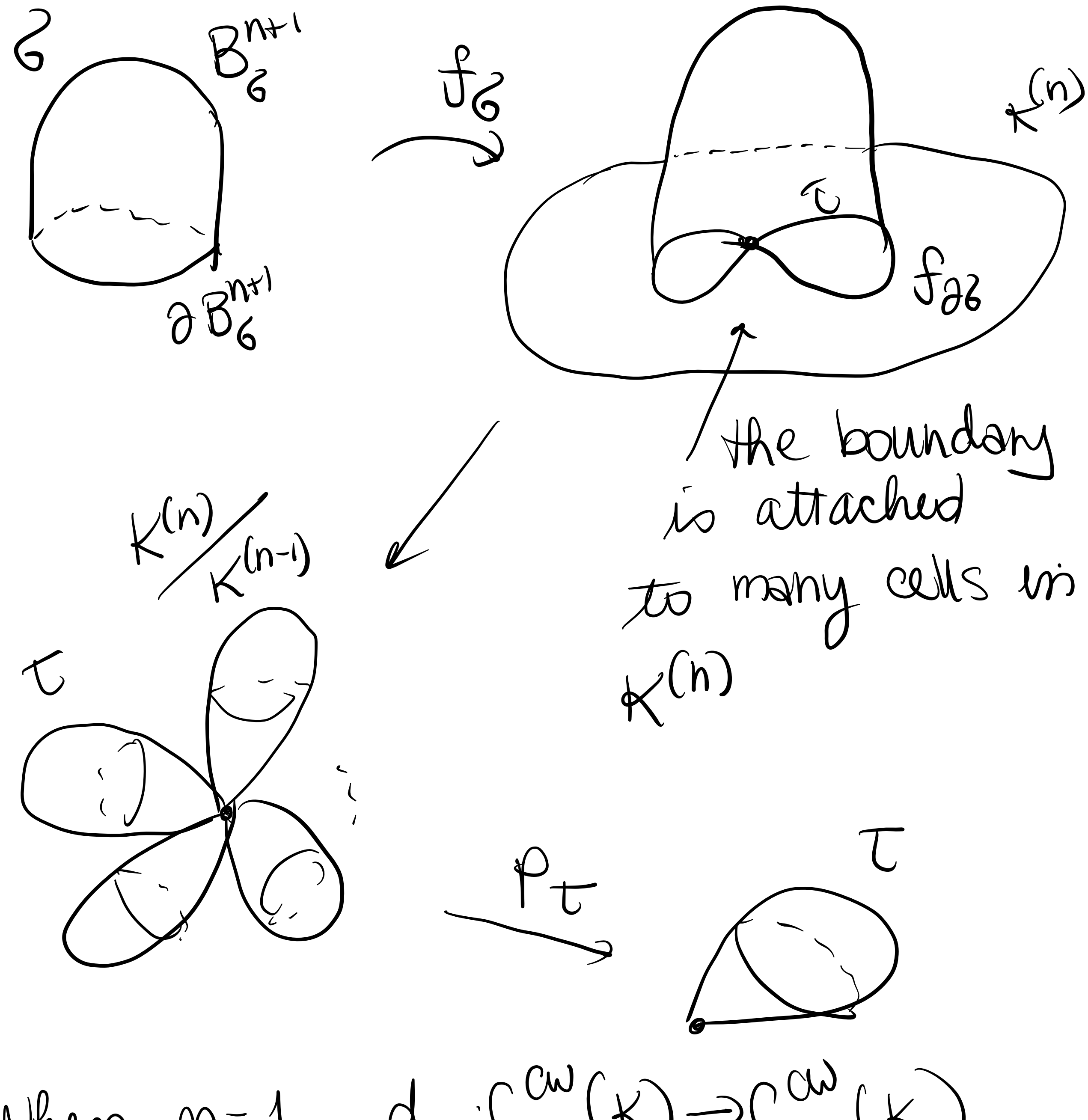
ϕ_n tells what multiple of the generator for τ we take

simplified notation

$$= \sum_\tau \deg (p_\tau \circ f_{\partial\sigma}) \cdot \tau$$

So, $[\tau : \sigma] = \deg (p_\tau \circ f_{\partial\sigma})$

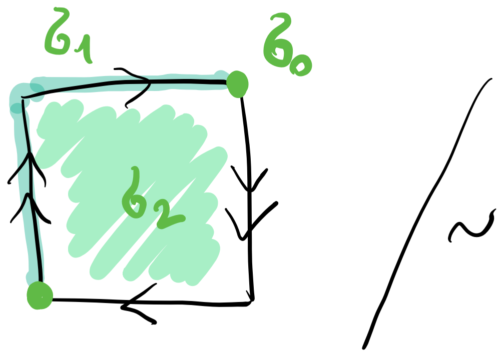
What is happening geometrically?



When $n=1$, $d_1 : C_1^{cw}(K) \to C_0^{cw}(K)$ is the same as the simplicial boundary map $\Delta_1(K) \to \Delta_0(K)$.

EXAMPLE

Take $K = \mathbb{R}P^2$. This is a CW-complex:



one 0-cell σ_0

one 1-cell σ_1

one 2-cell σ_2

CW-chain complex

$$\cdots \to 0 \to \mathbb{Z}\sigma_2 \to \mathbb{Z}\sigma_1 \to \mathbb{Z}\sigma_0 \to 0$$

$d\sigma_0 = 0$

$d\sigma_1 = 0$ ← starting & ending point are the same

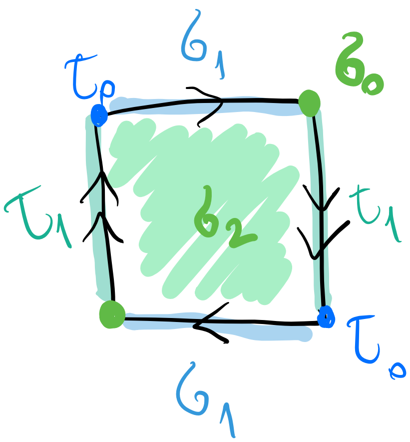
$d\sigma_2 = 2\cdot\sigma_1$ (or $-2\sigma_1$) sign depends on the choice of generators

$H_0(\mathbb{R}P^2) \cong \mathbb{Z}$

$H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$

$H_2(\mathbb{R}P^2) \cong 0$

Option #2 CW structure



0-cells

τ_0, σ_0

1-cells

τ_1, σ_1

2-cells

σ_2

Cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^2 \to 0$$

$$d_2(\sigma_2) = 2\tau_1 + 2\sigma_1 = 2(\tau_1 + \sigma_1)$$

$$d_1(\tau_1) = \tau_0 - \sigma_0 \qquad \ker d_1 = \langle \tau_1 + \sigma_1 \rangle$$

$$d_1(\sigma_1) = \sigma_0 - \tau_0$$

$$H_0(\mathbb{R}P^2) = \langle \tau_0, \sigma_0 \rangle / \operatorname{Im} d_1 \cong \langle \tau_0 - \sigma_0, \sigma_0 \rangle / \langle \tau_0 - \sigma_0 \rangle \cong \mathbb{Z}$$

$$H_1(\mathbb{R}P^2) = \langle \tau_1 + \sigma_1 \rangle / 2\langle \tau_1 + \sigma_1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_2(\mathbb{R}P^2) = \ker d_2 = 0$$

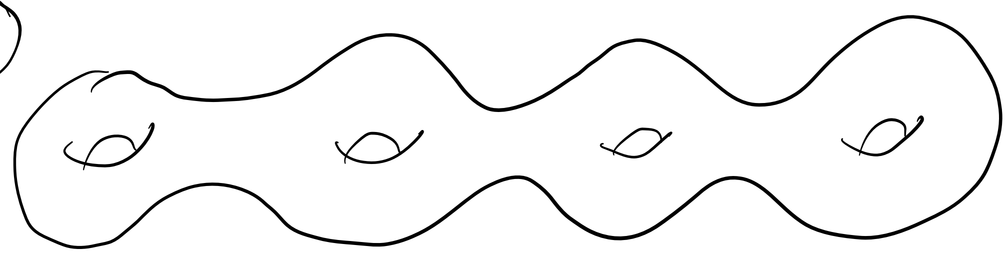
When doing exercises you don't have to worry about finding minimal cw-structure.

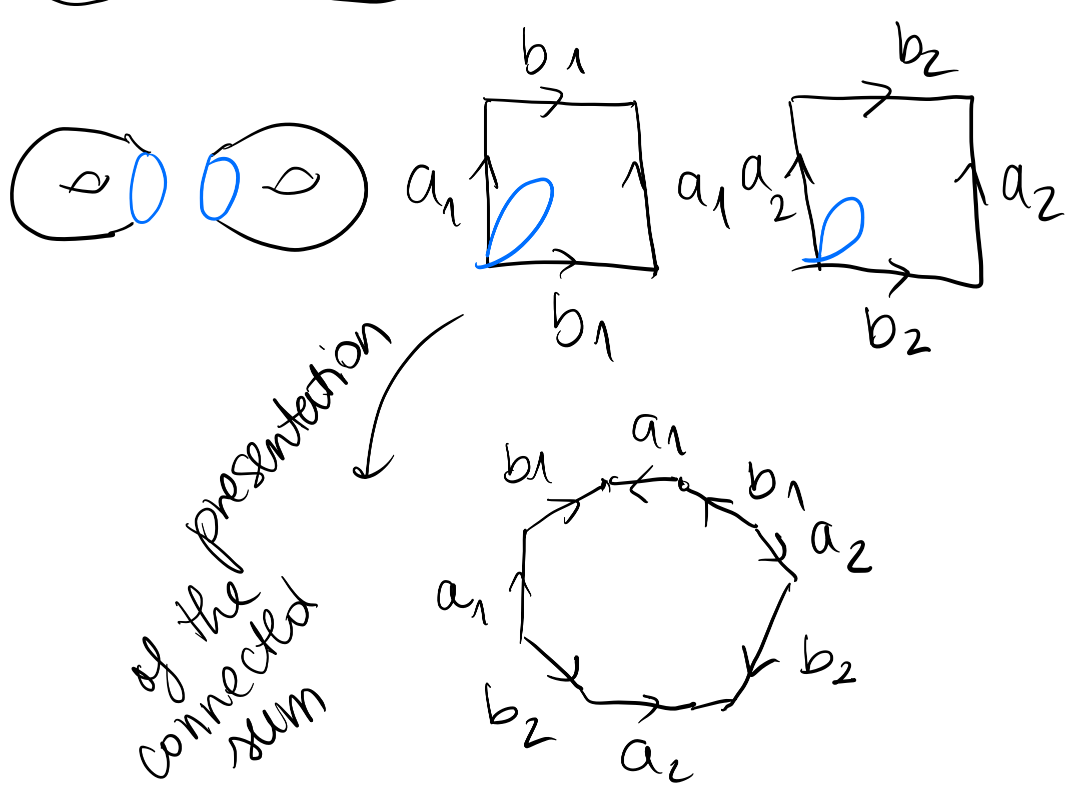
EXERCISE

Let M_g be a closed orientable surface of genus g (connected sum of g many tori).

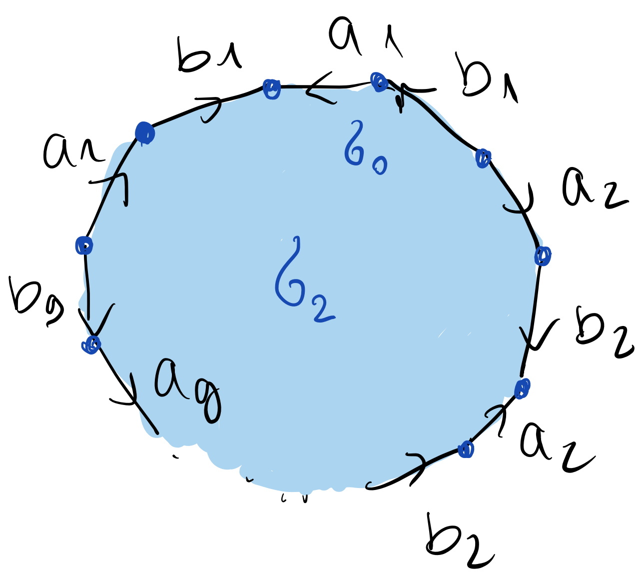
1. Find CW-structure on M_g.
2. Compute homology groups of M_g.

1.





For M_g we have the following representation



CW-structure

1 0-cell σ_0

$2g$ 1-cells

$a_1, a_2, b_1, b_2, \dots a_g, b_g$

1 2-cell σ_2

② Cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

$$d_2(\sigma_2) = a_1 + b_1 - a_1 - b_1 + a_2 + b_2 - a_2 - b_2 + \dots$$
$$\dots + a_g + b_g - a_g - b_g = 0$$

$$d_1(a_i) = 0, \quad d_1(b_i) = 0$$

So,

$$H_0(M_g) \cong \mathbb{Z}, \quad H_1(M_g) \cong \mathbb{Z}^{2g}, \quad H_2(M_g) \cong \mathbb{Z}.$$

WINTER 2016 Exam X, Y CW-complexes

ⓐ $k \in \mathbb{N}$, w $(k-1)$-cell σ $(k+1)$-cell

Show that

$$\sum_{\tau \text{ k-cells}} [w:\tau][\tau:\sigma] = 0$$

Proof

$$d_{n+1}(\sigma) = \sum_{\tau} [\tau:\sigma]\tau$$

$$0 = d_n \circ d_{n+1}(\sigma) = \sum_{\tau} [\tau:\sigma]\, d_n(\tau)$$

$$= \sum_{\substack{\tau \\ k\text{-cells}}} \sum_{\substack{w \\ (k-1)\text{-cells}}} [\tau:\sigma][w:\tau]\, w =$$

$$= \sum_{\substack{w \\ (k-1)\text{-cells}}} \left(\sum_{\substack{\tau \\ k\text{-cells}}} [\tau:\sigma][w:\tau] \right) w$$

$$\Rightarrow \sum_{\substack{\tau \\ k\text{-cells}}} [\tau:\sigma][w:\tau] = 0$$